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Effect of Disorder on Relaxation in the One-Dimensional Glauber Model

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We study the long-time relaxation of magnetization in a disordered linear chain of Ising spins from an initially aligned state. The coupling constants are ferromagnetic and nearest-neighbor only, taking values J_0 and J_1 with probabilities p and $1 - p$, respectively. The time evolution of the system is governed by the Glauber master equation. It is shown that for large times t, the magnetization $M(t)$ varies as $[\exp(-\lambda_0 t)]\Phi(t)$, where λ_0 is a function of the stronger bond strength J_0 only, and $\Phi(t)$ decreases slower than an exponential. For very long times, we find that $\ln \Phi(t)$ varies as $-t^{1/3}$. For low enough temperatures, there is an intermediate time regime when $\ln \Phi(t)$ varies as $-t^{1/2}$. The results can be extended to more general probability distributions of ferromagnetic coupling constants, assuming that $M(t)$ can only increase if any bond in the chain is strengthened. If the coupling constants have a continuous distribution in which the probability density varies as a power law near some maximum value J_0, we find that $\ln \Phi(t)$ varies as $-t^{1/3}(\ln t)^{2/3}$ for large times.

KEY WORDS: Relaxation; disorder; Ising chain; Glauber model.

1. INTRODUCTION

Quenched disorder can have important effects on the time-dependent properties of magnetic systems and on the approach to equilibrium in particular. In this paper, we study how the magnetization relaxes to equilibrium in a simple model of a disordered magnet. The model consists of a linear chain of Ising spins with random exchange interactions evolving in time as in the Glauber model.[1] Glauber wrote down a master equation to describe the time evolution of an Ising chain and studied the case when all exchange interactions were equal. He found that the uniform magnetization decays exponentially from an initially aligned state with a temperature-dependent relaxation

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time. In our case, with random coupling constants, the system has no translational invariance, and the uniform magnetization mode is not an eigenmode of the time evolution operator. The decay of the magnetization is not exponential any longer, and we determine its long-time behavior for various probability distributions of the exchange interaction.

In mathematical terms, the problem reduces to that of finding the spectral properties of a tridiagonal matrix with random elements. In that sense, it is quite similar to the problem of finding the density of states of a disordered harmonic chain[2–7] or of electrons in a one-dimensional disordered medium.[6] However, there are important differences. For one thing, the matrix in our case is nonsymmetric, unlike the Hamiltonian matrix in the vibrational or electronic problems. For another, the matrix elements involve two exchange couplings at a time, and so are not independent random variables. Also, the uniform mode is not an eigenmode of the matrix, as it is in the vibrational problem.[7]

A concept originally developed for the vibrational and electronic problems which proves useful in the magnetic relaxation case as well is the idea of fluctuation states.[8,9] In the two-mass vibrational problem, fluctuation states are due to the occurrence of large clusters of the lighter isotope. Such clusters have modes with frequencies very close to the maximum possible frequency; how close they are depends on how large the cluster is. They contribute to the density of states in the tail end of the band. The arguments of Lifshitz applied to one dimension give the density of states

$$\rho(E) \sim \exp(-A|E - E_0|^{-1/2}) \tag{1}$$

where E_0 is the band edge energy. In the magnetic case, fluctuation states arise from configurations in which there are large clusters of strong bonds. The larger a cluster of strong bonds, the slower it relaxes, and so at long times, the magnetization is dominated by large clusters. Lifshitz's estimate (1) is actually a lower bound on $\rho(E)$. In our problem, we are able to obtain a coinciding upper bound and so show that (1) is the right answer, and determine the constant A.

The plan of the paper is as follows: In Section 2, we formulate the problem and reduce it to diagonalizing a random matrix. We show how the long-time behavior of the magnetization depends on the properties near the band edge of a function $D(E)$, which is quite similar to $\rho(E)$ discussed above.

In Section 3, we study the case of a two-peaked probability distribution of the exchange, with one of the peaks at zero. The chain then splits up into noninteracting segments of various lengths, and the magnetization can be found by averaging over segments. The results at low temperatures depend on the relative magnitudes of ξ_T and ξ_P, which are two correlation lengths in the problem. ξ_T is the value of the thermal correlation length in an infinite

pure chain, whereas ξ_P is the percolation correlation length—essentially the average size of a segment. We show that if $\xi_P \gg \xi_T$, the magnetization $M(t)$ behaves as $\exp(-\lambda_0 t - ct^{1/3})$ at long times ($t \gg \xi_P^2$). If, on the other hand, $\xi_T \gg \xi_P$, there is an interesting crossover behavior. For $\xi_P \xi_T \ll t \ll \xi_T^3/\xi_P$, we find $M(t) \sim \exp(-\lambda_0 t - bt^{1/2})$, whereas when $t \gg \xi_T^3/\xi_P$, the magnetization goes as $\exp(-\lambda_0 t - ct^{1/3})$. We determine λ_0, c, and b in terms of the parameters in the model.

We turn to the case of a two-peaked distribution in which neither of the peaks is at zero in Section 4. We use the node counting theorem to derive a Lifshitz-type lower bound on the density of eigenvalues near the edge. We also derive an upper bound which coincides asymptotically with the lower bound. We find that the qualitative behavior of $M(t)$ in this case is the same as in Section 3.

In Section 5, we turn to arbitrary distributions of the exchange interactions. Assuming that increasing a bond strength can only slow down the relaxation, we use a bond strengthening (weakening) procedure together with the results of Sections 3 and 4 to derive upper (lower) bounds on the magnetization, and thus determine its asymptotic behavior for large times.

2. EQUIVALENCE OF THE PROBLEM TO DIAGONALIZATION OF A RANDOM MATRIX

We consider a linear chain of Ising spins σ_i ($i = 1$ to N) described by the Hamiltonian

$$H = -\sum_{i=1}^{N-1} J_{i+1/2}\sigma_i\sigma_{i+1} \tag{2}$$

The coupling constants $J_{i+1/2}$ ($i = 1$ to $N-1$) are quenched, independent, identically distributed random variables whose distribution function is given by

$$P(J) = p\,\delta(J - J_1) + q\,\delta(J - J_0) \tag{3}$$

Here $q = 1 - p$ and $0 \leqslant J_1 < J_0$. It is convenient to define a parameter ϵ as

$$\epsilon = -\ln q \tag{4}$$

We will consider more general distributions than that given by Eq. (3) in Section 5.

In the Glauber model[1] the time evolution of the system is assumed to be governed by the master equation

$$\frac{d}{dt}\mathscr{P}(\{\sigma_1 \cdots \sigma_i \cdots \sigma_N\}) = \sum_i [W_i^+ \mathscr{P}(\{\sigma_1 \cdots -\sigma_i \cdots \sigma_N\}) - W_i^- \mathscr{P}(\{\sigma_1 \cdots \sigma_i \cdots \sigma_N\})] \tag{5}$$

where $\mathscr{P}$ is the probability that the spin configuration is $\{\sigma_1 \cdots \sigma_i \cdots \sigma_N\}$ at time t, and W_i^+ and W_i^- are spin-flip probabilities per unit time for the spin i. The choice[10]

$$W_i^{\pm} = (1/2\tau)[1 \pm \sigma_i \tanh \beta h_i(t)] \tag{6}$$

is consistent with the detailed balance condition, and leads to the equation

$$(1 + \tau\, \partial/\partial t)\langle \sigma_i(t)\rangle = \langle \tanh \beta h_i(t)\rangle \tag{7}$$

$h_i(t)$ is the instantaneous effective field at site i at time t, given by

$$h_i(t) = J_{i-1/2}\sigma_{i-1}(t) + J_{i+1/2}\sigma_{i+1}(t) \tag{8}$$

and $\langle \cdots \rangle$ denotes an average over $\mathscr{P}$. Using the fact that each σ_i takes values ± 1 only, we can write Eq. (7) as

$$(1 + d/dt)S_i(t) = C_i^- S_{i-1}(t) + C_i^+ S_{i+1}(t) \tag{9}$$

where we have chosen units of time so that $\tau = 1$, and defined

$$S_i(t) = \langle \sigma_i(t)\rangle \tag{10}$$

$$C_i^{\pm} = \tfrac{1}{2}[\tanh \beta(J_{i+1/2} + J_{i-1/2}) \pm \tanh \beta(J_{i+1/2} - J_{i-1/2})] \tag{11}$$

Equations (8)–(11) hold for the boundary spins $i = 1$ and N also if we define $J_{1/2}, J_{N+1/2} \equiv 0$. Note that Eqs. (9) form a closed set of equations describing the time evolution of $S_i(t)$, and we do not need higher order correlation functions. This would not be true in higher dimensions.

Let us assume that at $t = 0$, all spins are aligned parallel and up. In time, they will relax to the equilibrium state with zero magnetization. The average magnetization at time t is given by

$$M(t) = (1/N) \sum_{i=1}^{N} \langle S_i(t)\rangle_c \tag{12}$$

where $\langle \cdots \rangle_c$ denotes averaging over the quenched variables $\{J_{i+1/2}\}$. We are interested in determining the behavior of $M(t)$ for large times $t \gg 1$.

Equation (9) may be written as a vector equation determining the time evolution of a vector $|S(t)\rangle$ whose ith component is $(1/\sqrt{N})S_i(t)$. The equation is of the form

$$\frac{d}{dt}|S(t)\rangle = -\Lambda|S(t)\rangle \tag{13}$$

Here Λ is an $N \times N$ matrix independent of time, and from Eq. (9) it is clear that it is nonsymmetric, tridiagonal, and real. Equation (13) has the solution

$$|S(t)\rangle = \exp(-\Lambda t)\,|S(0)\rangle = \sum_{\lambda} \exp(-\lambda t)\,|e_{\lambda,R}\rangle\langle e_{\lambda,L}|S(0)\rangle \tag{14}$$

where

$$|S(0)\rangle = (1/\sqrt{N})\langle 1, 1, 1, \ldots, 1|^T \tag{15}$$

and λ, $|e_{\lambda,R}\rangle$, and $\langle e_{\lambda,L}|$ denote, respectively, the eigenvalues and the right and left eigenvectors of Λ. Taking the configuration average, we get

$$M(t) = \int_0^\infty d\lambda\, D(\lambda) \exp(-\lambda t) \tag{16}$$

where

$$D(\lambda)\, d\lambda = \left\langle \sum_{\lambda \leqslant \lambda' \leqslant \lambda + d\lambda} \langle S(0)|e_{\lambda' R}\rangle \langle e_{\lambda' L}|S(0)\rangle \right\rangle_C \tag{17}$$

It is clear from Eq. (16) that to determine the behavior of $M(t)$ for large times, it is sufficient to determine the behavior of $D(\lambda)$ for λ near λ_0, the lowest allowed eigenvalue. The states that contribute to $D(\lambda)$ in this region are precisely the fluctuation states of Lifshitz, and we estimate their contribution in subsequent sections.

3. THE CASE $J_1 = 0$

We first consider the case when J_1 in Eq. (3) is set equal to zero. In this case, the chain breaks up into unconnected clusters of spins, and the time evolution of each cluster is independent of the others.

It is easy to see that the probability P_l that an arbitrarily chosen site belongs to a cluster containing l spins is given by

$$P_l = lp^2q^{l-1} \tag{18}$$

and the expected number of clusters of size l is NP_l/l.

The average magnetization at time t is given by

$$M(t) = \sum_{l=1}^{\infty} P_l M_l(t) \tag{19}$$

where $M_l(t)$ is the magnetization of a cluster of size l and is determined quite easily in terms of the eigenvalues and eigenvectors of a finite $l \times l$ tridiagonal matrix. The eigenvectors are cosines and sines, but the odd modes (sines) do not contribute to the sum in Eq. (17). The eigenvalues are found to be

$$\lambda_i = 1 - \tanh 2\beta J_0 \cos k_i \tag{20}$$

For the cosine modes, the k_i are the solutions to the equation

$$\tan k \tan[\tfrac{1}{2}(l-1)k] = a \tag{21}$$

where

$$a = \tanh 2\beta J_0 \coth \beta J_0 - 1 \tag{22}$$

We can think of each eigenvector of a finite cluster as defining a localized eigenmode for the entire chain, with $S_i = 0$ for all i outside the cluster. From the explicit expressions for left and right eigenvectors, it is straightforward to verify that for all eigenvalues λ

$$\langle S(0)|e_{\lambda,R}\rangle\langle e_{\lambda,L}|S(0)\rangle \geqslant 0 \tag{23}$$

If $|e_{\lambda,R}\rangle$ has the lowest eigenvalue among the modes localized at a particular cluster of size l, we have

$$\langle S(0)|e_{\lambda,R}\rangle\langle e_{\lambda,L}|S(0)\rangle \geqslant 4\pi^{-2}l/N \tag{24}$$

The allowed values of λ are bounded from below by λ_0, obtained by setting $k_i = 0$ in Eq. (20), i.e.,

$$\lambda \geqslant \lambda_0 = 1 - \tanh 2\beta J_0 \tag{25}$$

This implies that

$$D(\lambda) = 0 \qquad \text{if} \quad \lambda < \lambda_0 \tag{26}$$

Consider now a value of λ slightly greater than λ_0. Then there exists a critical value of l, say $l_c(\lambda)$, such that for all $l < l_c(\lambda)$, no eigenvalue of an l-cluster is below λ. From Eq. (21), we see immediately that

$$l_c(\lambda) = 1 + \lceil (2/k)\tan^{-1}(a \cot k)\rceil \tag{27}$$

where $\lceil x\rceil$ denotes the smallest integer not smaller than x. Here k is an implicit function of λ, defined by the equation

$$\lambda = 1 - \tanh 2\beta J_0 \cos k \tag{28}$$

Now any cluster of length $l \geqslant l_c(\lambda)$ has at least one eigenvalue less than λ, which, by Eq. (24), contributes at least $(2/\pi)^2 l/N$ to $D(\lambda)$. Defining

$$I(\lambda) = \int_{\lambda_0}^{\lambda} d\lambda'\, D(\lambda') \tag{29}$$

we get

$$I(\lambda) \geqslant \left(\frac{2}{\pi}\right)^2 \sum_{l=l_c}^{\infty} \frac{l}{N} \times (\text{expected number of clusters of length } l)$$

i.e.,

$$I(\lambda) \geqslant (2/\pi)^2 q^{l_c(\lambda)-1} \tag{30}$$

Thus, $I(\lambda)$ is nonzero for $\lambda > \lambda_0$. As λ approaches λ_0 from above, $l_c(\lambda)$ tends to infinity as $(\pi/\sqrt{2})(\lambda - \lambda_0)^{-1/2}$, and (30) becomes

$$I(\lambda) \geqslant (2/\pi)^2 \exp[-A(\lambda - \lambda_0)^{-1/2}] \tag{31}$$

where

$$A = \epsilon\pi/\sqrt{2} \tag{32}$$

We now derive an upper bound for $I(\lambda)$. If $l > l_c(\lambda)$, at least one eigenvalue of the l-cluster falls below λ, and if it is large enough, more than one falls below λ. From Eq. (23), each additional mode makes a positive contribution to $I(\lambda)$. But by completeness, the sum of the contribution of modes localized at one l-cluster is at most (l/N). Thus

$$I(\lambda) \leqslant q^{l_c(\lambda)-1} \tag{33}$$

The bounds (30) and (33) differ only by a multiplicative factor, and determine the asymptotic behavior of $I(\lambda)$ [and thus $D(\lambda)$] for small $(\lambda - \lambda_0)$ fairly accurately. In the following, we shall write

$$D(\lambda) \approx A(\lambda) \exp\left[-\frac{2\epsilon}{k} \tan^{-1} \frac{a}{k}\right] \qquad \text{as } \lambda \to \lambda_0 \text{ from above} \tag{34}$$

Here $A(\lambda)$ is some unspecified, slowly varying function of λ. Note that some smearing out of $D(\lambda)$ is implicit in an analytic form such as that in Eq. (34), as the actual function $D(\lambda)$ is a sum of a denumerably infinite number of delta functions of varying strengths.[6] The exact form of $A(\lambda)$ will not matter in the following arguments.

Substituting Eq. (34) in Eq. (16), we can find $M(t)$. For large t, the integral in Eq. (16) can be evaluated with asymptotic exactness by the method of steepest descent, and we get

$$\ln M(t) \approx \max_{\lambda_0 \leqslant \lambda \leqslant \infty} [-\lambda t + \ln D(\lambda)] \tag{35}$$

It is convenient to separate out the purely exponential part of $M(t)$, and define $\Phi(t)$ by the equation

$$M(t) = [\exp(-\lambda_0 t)]\Phi(t) \tag{36}$$

$\Phi(t)$ decays less rapidly than exponentially, and we find it below. For large β and small k, we may write

$$\lambda(k) \cong 1 - \tanh 2\beta J_0 + k^2/2 \tag{37}$$

From Eqs. (35)–(37), we find

$$\ln \Phi(t) \cong \max_k [-\tfrac{1}{2}tk^2 - (2\epsilon/k) \tan^{-1}(a/k)] \tag{38}$$

If we define the variables

$$\tilde{t} = ta^3/\epsilon, \qquad \tilde{k} = k/a \tag{39}$$

we can rewrite this as

$$\ln \Phi(t) = -(\epsilon/a)\tilde{F}(\tilde{t}) \tag{40}$$

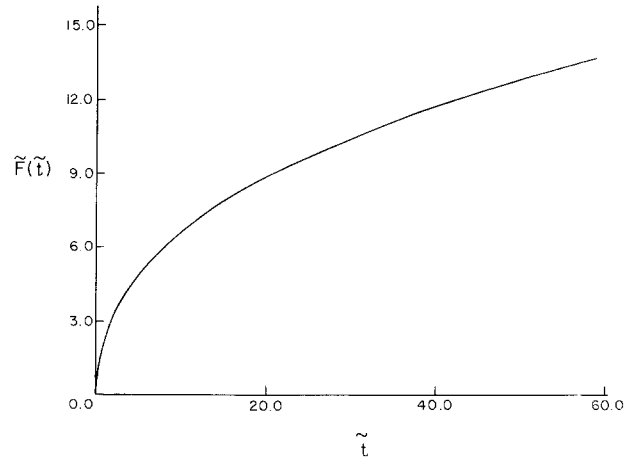

Fig. 1. Graph of $\tilde{F}(\tilde{t})$ vs. $\tilde{t}$, showing the crossover from the $\tilde{t}^{1/2}$ behavior for $\tilde{t} \ll 1$ to the $\tilde{t}^{1/3}$ behavior for $\tilde{t} \gg 1$.

where $\tilde{F}(\tilde{t})$ is defined by

$$\tilde{F}(\tilde{t}) = \underset{0 \leqslant k \leqslant \infty}{\text{Min}} [\tfrac{1}{2}\tilde{t}\tilde{k}^2 + (2/\tilde{k}) \tan^{-1}(1/\tilde{k})] \tag{41}$$

Figure 1 shows a graph of $\tilde{F}(\tilde{t})$. For small $\tilde{t}$, the extremizing value of $\tilde{k}$ in Eq. (41) is large, and

$$\tilde{F}(\tilde{t}) \approx \underset{0 \leqslant k \leqslant \infty}{\text{Min}} [\tfrac{1}{2}\tilde{t}\tilde{k}^2 + 2/\tilde{k}^2] = 2\tilde{t}^{1/2} \tag{42}$$

For large $\tilde{t}$, the extremizing $\tilde{k}$ is small and

$$\tilde{F}(\tilde{t}) \approx \underset{0 \leqslant k \leqslant \infty}{\text{Min}} [\tfrac{1}{2}\tilde{t}\tilde{k}^2 + \pi/\tilde{k}] = \tfrac{3}{2}\pi^{2/3}\tilde{t}^{1/3} \tag{43}$$

The steepest descent approximation cannot be valid for arbitrarily small times, as is also evident from the fact that Eq. (42) has an unphysical singularity at $\tilde{t} = 0$. The cutoff is given by the requirement that $(\epsilon/a)\tilde{F}(\tilde{t}) \gg 1$.

In order to summarize our results in a more transparent manner, let us define thermal and percolation correlation lengths ξ_T and ξ_P, respectively, by

$$\xi_T = -(\ln \tanh \beta J_0)^{-1} \tag{44}$$

$$\xi_P = 1/\epsilon \tag{45}$$

For large values of β,

$$\xi_T \approx 1/a \approx \exp(2\beta J_0)/2 \tag{46}$$

We are primarily interested in what happens when both ξ_P and ξ_T are large.

If $\xi_P \gg \xi_T$, then broken bonds have no very significant effect on the relaxation, except at very large times, when $M(t)$ has an additional $\exp(-t^{1/3})$ correction, i.e.,

$$\ln M(t) \approx -t/(2\xi_T^2) - \tfrac{3}{2}\pi^{2/3}\xi_P^{-2/3}t^{1/3} \qquad \text{for } t \gg \xi_P^2 \tag{47}$$

If, however, the temperature is so low that $\xi_T \gg \xi_P$, then for a long time $\Phi(t)$ is of the form $\exp(-t^{1/2})$, but ultimately crosses over to $\exp(-t^{1/3})$. Explicitly,

$$\ln M(t) \approx -t/(2\xi_T^2) - 2\xi_T^{-1/2}\xi_P^{-1/2}t^{1/2} \qquad \text{for } \xi_T\xi_P \ll t \ll \xi_T^3/\xi_P \tag{48}$$

and

$$\ln M(t) \approx -t/(2\xi_T^2) - \tfrac{3}{2}\pi^{2/3}\xi_T^{-2/3}t^{1/3} \qquad \text{for } t \gg \xi_T^3/\xi_P \tag{49}$$

The crossover time ξ_T^3/ξ_P may be obtained by equating the two expressions for $M(t)$ in (48) and (49). Note that these "corrections" to the purely exponential relaxation of $M(t)$ are quite important; in fact, for $t < \xi_T^3/\xi_P$, the "correction term" is larger than the linear term $(t/2\xi_T^2)$ in $\ln M(t)$.

This concludes our discussion of the case $J_1 = 0$.

4. THE CASE OF NONZERO J_1

In this section, we argue that even if $J_1 \neq 0$, the function $D(\lambda)$ has the same asymptotic form as in Eq. (34) for λ close to λ_0. The case of $J_1 \neq 0$ only changes the value of the constant a appearing in that equation. We determine upper and lower bounds for a in this case.

Let $F(\lambda)$ be the fractional number of eigenvalues of Λ that are less than λ. Thus $F(\lambda)$ is the integrated spectral density for the disordered chain, with bond strengths distributed according to Eq. (3). We show below that

$$F(\lambda) \geqslant pq^{n_H} \tag{50}$$

and

$$F(\lambda) \leqslant q^{n_L}(1 + p^{n_L} + p - pq^{n_L}) \tag{51}$$

where n_H and n_L are integers which depend on λ. For small values of λ, their values are given approximately by

$$n_H \approx \lceil (2/k)\tan^{-1}(a_H/k) \rceil \tag{52}$$

and

$$n_L \approx \lceil (2/k)\tan^{-1}(a_L/k) \rceil \tag{53}$$

Here a_L and a_H are some constants which depend on βJ_0 and βJ_1, and k is defined in terms of λ by Eq. (28) as before. Explicit expressions for a_L and a_H

are given later. For $\lambda \to \lambda_0$, both the upper and lower bounds on $F(\lambda)$ have the same asymptotic behavior, and suggest that we write

$$F(\lambda) \approx A(\lambda) \exp[-(2\epsilon/k) \tan^{-1}(a/k)] \tag{54}$$

where $A(\lambda)$ is some slowly varying function of λ, and we have

$$a_L \leqslant a \leqslant a_H \tag{55}$$

We will show that a must decrease exponentially fast with β for large β, as both a_L and a_H do so.

Let $G(\lambda)$ be the average value of the matrix element product $\langle S(0)|e_{\lambda R}\rangle\langle e_{\lambda L}|S(0)\rangle$ averaged over eigenmodes with eigenvalues between λ and $\lambda + d\lambda$. We expect that $G(\lambda)$ is a slowly varying function of λ, varying at most like $(\lambda - \lambda_0)^{-x}$ for some small, positive x. This function can be absorbed in $A(\lambda)$, and from Eq. (54) we get

$$D(\lambda) \approx A(\lambda) \exp[-(2\epsilon/k) \tan^{-1}(a/k)] \tag{56}$$

This is of the same form as Eq. (34), and the behavior of $M(t)$ for large t can be found exactly as before. Again we find an $\exp(-ct^{1/3})$ correction to the exponential decay for $M(t)$ for very long times. If we go to low enough temperatures, a well-defined window in time develops (what may be called the "short" long-time regime, corresponding to $\tilde{t} \ll 1$) when the correction is $\exp(-bt^{1/2})$.

We now derive the bounds given in Eqs. (50)–(53).

Let us introduce two dummy spins S_0 and S_{N+1}, which are coupled to S_1 and S_N, respectively, with infinitesimal positive bond strengths $J_{1/2} = J_{N+1/2} \cong 0$. We consider a right eigenvector of Λ with eigenvalue λ and components $S_i(\lambda)$. We further define

$$\xi_i(\lambda) = S_i(\lambda)/S_{i-1}(\lambda) \tag{57}$$

$$\xi_1 = \infty \tag{58}$$

It follows from Eq. (9) that the ξ_i satisfy the recurrence relations

$$\xi_{i+1} = [(1 - \lambda) - C_i^-/\xi_i]/C_i^+ \tag{59}$$

These recurrence relations can be used to determine successive values of the ξ_i for any given λ and a given configuration of bond strengths on the chain. The values $\{\xi_i\}$ define an eigenmode of the chain if λ is such that ξ_{N+1} is zero. A negative value of ξ_i implies that $S_i(\lambda)$ and $S_{i-1}(\lambda)$ have opposite signs, i.e, the eigenmode has a node between the sites i and $i - 1$. Using the facts that $C_i^{\pm} \geqslant 0$, it can be shown that all the eigenvalues of Λ are real, and the number of negative ξ_i corresponding to any λ is equal to the number of eigenvalues of Λ below λ. This is the node-counting theorem[4–6] for our problem.

Let the sites $i+1, i+2, \ldots, i+n$ form a cluster of size n, which means that $J_{i+1/2} = J_{i+n+1/2} = J_1$, and all the intermediate bonds are of strength J_0. Consider the propagation of a trial mode of eigenvalue λ along this cluster. The recurrence relations (59) uniquely determine ξ_{i+n+1}, given ξ_{i+1}; and we may write

$$\xi_{i+n+1} = f_n(\xi_{i+1}) \tag{60}$$

These f_n will be called cluster transfer functions. The propagation of a mode along the chain may be viewed as successive applications of these cluster transfer functions on the "initial state" ξ_1. The value of ξ after transfer across r clusters of lengths $n_1, n_2, \ldots, n_r$ is $f_{n_r} \cdots f_{n_2} f_{n_1}(\xi)$. Since the lengths of adjacent clusters are independent random variables, we may think of these transfer functions as evolution operators for a Markov process.

From the linearity of Eq. (9), it is easy to see that the functions f_n are of the form

$$f_n(\xi) = U_n - V_n/(\xi - W_n) \tag{61}$$

Here U_n, V_n, and W_n are functions of J_0, J_1, and λ. Explicit expressions for these may be written down quite easily.

As we propagate a mode across an n-cluster, the solution may or may not have any nodes inside the n-cluster, depending on λ and n and the value of ξ just before entering the cluster. (If a node falls between two clusters, we shall assign it to the cluster closer to the node. For example, if sites i and $i+1$ belong to different clusters and ξ_{i+1} is negative, the node is assigned to the cluster containing site i if $\xi_{i+1} < -1$; otherwise it is assigned to the cluster containing the site $i+1$.)

It is quite clear that there exists a minimum value of n, say $n_H(\lambda)$, such that for $n > n_H(\lambda)$ and for all values of ξ just before entering the cluster, an n-cluster has at least one node for a propagating mode of eigenvalue λ. This is trivially true if $n > \pi/k$. A better estimate of n_H is obtained by finding the largest value of r such that for all $n \leqslant r$

$$f_n(-1) > -1 \tag{62}$$

Negation of this condition would imply that there exists a mode with two nodes inside the r-cluster. But two independent eigensolutions of Eq. (9) with the same eigenvalue have interlacing zeros, and hence any other solution of eigenvalue λ will have at least one node inside the r-cluster (Fig. 2). From Eq. (62), the value of n_H is easily determined. The solid curve in Fig. 2 is symmetric about the middle of the cluster, and by Eq. (9) it must have a cosine dependence on the site labels i within the n-cluster. Hence we write

$$S_j(\lambda) = A \cos k\left(j - i - \frac{n+1}{2}\right) \qquad \text{for } i+1 \leqslant j \leqslant i + n_H + 1 \tag{63}$$

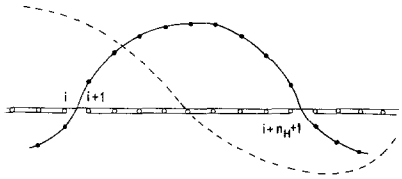

Fig. 2. Graph of a mode with two nodes just outside an n-cluster. We represent the strong bonds by double lines joining the corresponding sites. The dashed curve is a mode with the same eigenvalue as the mode shown by the solid curve, but for which the value of ξ on entering the cluster is not -1.

and

$$S_i(\lambda)/S_{i+1}(\lambda) \geqslant -1 \tag{64}$$

The requirement that Eq. (9) is satisfied at site $(i+1)$, together with (64) with the inequality sign replaced by equality, gives us the value of n_H. We find that $n_H = \lceil l \rceil - 1$, where l is the smallest positive solution of the equation

$$\{1 - \lambda + \tfrac{1}{2}[\tanh \beta(J_0 + J_2) - \tanh \beta(J_0 - J_1)]\} \cos[\tfrac{1}{2}(l-1)k]$$
$$= \tfrac{1}{2}[\tanh \beta(J_0 + J_1) + \tanh \beta(J_0 - J_1)] \cos[\tfrac{1}{2}(l-3)k] \tag{65}$$

For small k, this equation simplifies to

$$k \tan[\tfrac{1}{2}(l-1)k] \approx a_H \tag{66}$$

where

$$a_H = 2[\tanh 2\beta J_0 - \tanh \beta(J_0 - J_1)][\tanh \beta(J_0 + J_1) + \tanh \beta(J_0 - J_1)]^{-1} \tag{67a}$$

$$\approx 2 \exp(-2\beta J_0 + 2\beta J_1) \qquad \text{for large } \beta \tag{67b}$$

From Eq. (66) and the fact that $n_H = \lceil l \rceil - 1$ we get Eq. (52). Now the number of nodes in a mode corresponding to eigenvalue λ is certainly greater than the number of clusters of size greater than n_H. This gives (50).

The upper bound to $F(\lambda)$ is established similarly. We define $n_L(\lambda)$ as the largest integer satisfying the following conditions:

(i) For all $n \leqslant n_L(\lambda)$ and for all $\xi \geqslant 1$

$$f_n(\xi) \geqslant 1 \tag{68}$$

(ii) The corresponding solution has no nodes inside the n-cluster.

Consider a sequence of clusters of lengths $n_1, n_2, \ldots$, the length of each cluster in this sequence being less than or equal to n_L. If the value of ξ just before entering this sequence of clusters is greater than one, it would stay

greater than one after transfer across the first, second,..., clusters, according to (68). Hence this mode will have no modes as it propagates along this sequence of clusters. Again, by the interlacing property of zeros of modes with equal eigenvalues, any other mode of eigenvalue λ propagating along this sequence of clusters can have at most one node within this sequence.

If n_L is large, there are very long sequences of the above type in a typical configuration of the disordered chain. These are interrupted by a few clusters or groups of clusters of sizes greater than n_L. Any of these n-clusters ($n > n_L$) can have at most n nodes. Let Nf_s be the expected number of sequences of short clusters which are separated from each other by long clusters of sizes greater than n_L. We get an upper bound to the number of nodes corresponding to eigenvalue λ in terms of f_s as

$$F(\lambda) \leqslant \left[\sum_{l=n_L+1}^{\infty} P_l\right] + f_s \tag{69}$$

Working out f_s in terms of p, q, and n_L, we get

$$F(\lambda) \leqslant q^{n_L}(1 + n_L p + p - pq^{n_L})$$

An explicit expression for n_L as a function of λ is obtained by solving (68) with the inequalities replaced by equalities. The resulting equations are similar to (63) and (64), with the -1 in (64) replaced by $+1$. We find that $n_L = [l]$, where l is determined by the equation

$$\{2(1 - \lambda) - [\tanh \beta(J_0 + J_1) - \tanh \beta(J_0 - J_1)]\} \cos[\tfrac{1}{2}(l - 1)k]$$
$$= [\tanh \beta(J_0 + J_1) + \tanh \beta(J_0 - J_1)] \cos[\tfrac{1}{2}(l - 3)k] \tag{70}$$

For small k, this equation may be simplified to

$$k \tan[\tfrac{1}{2}(l - 1)k] = a_L \tag{71}$$

where

$$a_L = 2[\tanh 2\beta J_0 - \tanh \beta(J_0 + J_1)][\tanh \beta(J_0 + J_1) + \tanh \beta(J_0 - J_1)]^{-1} \tag{72a}$$

$$\approx 2 \exp(-2\beta J_0 - 2\beta J_1) \qquad \text{for large } \beta \tag{72b}$$

Equation (53) follows from Eq. (71) directly by requiring that $n_L = [l]$.

This completes our derivation of lower and upper bounds for $F(\lambda)$. Notice that for $J_1 = 0$ we have $a_L = a_H = a$. Also, note that both a_L and a_H decrease exponentially with β for large β.

It is possible to obtain better bounds on a by considering groups of clusters instead of single clusters treated above. For example, consider two adjacent clusters of sizes n_1 and n_2. Then it is easy to determine values of n_1 and n_2 such that a propagating mode of eigenvalue λ has at least one node

inside each double cluster (n_1, n_2). Counting the number of such double clusters in a long chain gives us a better estimate of $F(\lambda)$ than (50). Similarly, one can identify pairs (n_1, n_2) such that an infinite chain obtained by repeating the double cluster (n_1, n_2) has its lowest eigenvalue greater than λ. This will include all pairs with $n_1 \leqslant n_L$ and $n_2 \leqslant n_L$ and some more. Then, as argued earlier, an uninterrupted long sequence of such double clusters will have at most one node. This can be used to get an improved upper bound on $F(\lambda)$. The procedure may be generalized to triple clusters, etc. We do not derive these bounds as the calculation is tedious, but only remark that the resulting upper and lower bounds would be very close to each other. The most important configurations in these calculations are those in which a large cluster of strong bonds lies between two moderately large clusters of weak bonds.

5. OTHER PROBABILITY DISTRIBUTIONS AND THE FERROMAGNETIC INEQUALITY

In order to study more general distributions of bond strengths, we shall assume without proof the following generalized ferromagnetic inequality.

Conjecture. Consider a ferromagnetic kinetic Ising model in any dimension, specified by the Hamiltonian $H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j$, with $J_{ij} \geqslant 0$. The time evolution is governed by the Glauber master equation. The system need not be translationally invariant, and the number of spins may be finite or infinite. At time $t = 0$, suppose that all spins are aligned parallel and up. Then for all $t \geqslant 0$ and all sites i, j, and k, $S_i(t)$ is a monotonic, nondecreasing function of J_{jk}.

This is a generalization of the well-known GKS[11] inequalities to nonequilibrium situations. The conjecture is very plausible, and we would expect it to hold even if some (ferromagnetic) many-spin coupling terms are included in the Hamiltonian. A proof of the inequality in its full generality would be very desirable. We know of no counterexamples to the conjecture, but have been unable to devise a proof, even for our comparatively simple case of a one-dimensional chain with nearest-neighbor interactions. Some preliminary results which support the conjecture for the linear chain are given in the appendix.

Armed with the inequality, we can attack the problem of a general distribution of bond strengths.

Let J_0 be the maximum allowed value of J for some arbitrary ferromagnetic distribution $P(J)$ of bond strengths. Our first result for the average magnetization $M(t)$ is the following:

$$\lim_{t\to\infty}\{[\ln M(t)]/t\} = -(1 - \tanh 2\beta J_0) \equiv -\lambda_0 \tag{73}$$

This result shows that at large times the behavior of $P(J)$ for small J is irrelevant, and only the largest allowed bond strength is important.

This is proved quite easily as follows. Let us increase the strength of each bond in the chain to J_0, in which case the relaxation is purely exponential. By the ferromagnetic inequality, this can only increase the average magnetization $M(t)$, and so we have

$$M(t) \leqslant \exp[-t(1 - \tanh 2\beta J_0)] \tag{74}$$

We can also establish a similar lower bound. Choose any value of a bond strength J^* with $0 \leqslant J^* \leqslant J_0$. For any configuration of bonds $\{J_{i+1/2}\}$ we construct a new chain C' whose bond strengths are given by

$$J'_{i+1/2} = 0 \qquad \text{if} \quad J_{i+1/2} < J^* \tag{75a}$$

$$J'_{i+1/2} = J^* \qquad \text{if} \quad J_{i+1/2} \geqslant J^* \tag{75b}$$

$M'(t)$, the magnetization of C', decays faster than $M(t)$, and so

$$\operatorname*{Lim}_{t\to\infty}[t^{-1} \ln M(t)] \geqslant \operatorname*{Lim}_{t\to\infty}[t^{-1} \ln M'(t)] \tag{76}$$

But from Section 3, the right-hand side is equal to $-(1 - \tanh 2\beta J^*)$, and this holds for all choices of J^*. Choosing J^* arbitrarily close to J_0, we conclude that

$$\operatorname*{Lim}_{t\to\infty}[t^{-1} \ln M(t)] \geqslant -(1 - \tanh 2\beta J_0) \tag{77}$$

The relations (74) and (77) imply Eq. (73). This determines the leading behavior of $M(t)$ for large t.

We can also determine the asymptotic behavior of $\Phi(t)$. We have seen that only the structure of $P(J)$ for J close to J_0 is important for the long-time behavior of $M(t)$. Suppose that for J near J_0,

$$P(J) \sim (J_0 - J)^{\mu - 1} \tag{78}$$

where $\mu > 0$. We define $p'(J)$ by

$$1 - p'(J) = \int_J^{J_0} dJ' \, P(J') \tag{79a}$$

$$\approx K(J_0 - J)^{\mu} \tag{79b}$$

for small $(J_0 - J)$. Here K is some positive constant. $p'(J^*)$ is just the probability of broken bonds in the chain C'. From Section 3, the magnetization of C' for large times t is given by

$$\ln M'(t) \approx -(1 - \tanh 2\beta J^*)t - c\{\ln[1 - p'(J^*)]^{-1}\}^{2/3} t^{1/3} \tag{80}$$

We treat J^* as a variational parameter, and choose its value such that $M'(t)$ in Eq. (80) is as large as possible. This gives us the best lower bound to $M(t)$. A simple calculation shows that for large t, the optimal choice of J^* is

$$J^*(t) \approx J_0 - \alpha_1 t^{-2/3} \tag{81}$$

Substituting Eqs. (79b) and (81) in Eq. (80), we get

$$\ln M(t) \geqslant -\lambda_0 t - \alpha_1' t^{1/3} (\ln t)^{2/3} \tag{82}$$

Here a_1 and a_1' are some constants which depend on K and μ.

We can similarly derive a better upper bound on $M(t)$ than in Eq. (74). Consider a chain C'' whose bond strengths $\{J''_{i+1/2}\}$ are given by

$$J''_{i+1/2} = J^* \qquad \text{if} \quad J_{i+1/2} \leqslant J^* \tag{83a}$$

$$J''_{i+1/2} = J_0 \qquad \text{if} \quad J_{i+1/2} > J^* \tag{83b}$$

The ferromagnetic inequality then implies

$$M(t) \leqslant M''(t) \tag{84}$$

where $M''(t)$ is the magnetization of C''. The behavior of $M''(t)$ for large times t is known from Section 4 for any choice of J^*. It is given by

$$\ln M''(t) \approx -\lambda_0 t - c\{-\ln[1 - p'(J^*)]\}^{2/3} t^{1/3} \tag{85}$$

We would like to choose J^* to make $M''(t)$ as small as possible. This occurs when $p'(J^*)$ is as large as possible. However, if J^* is very close to J_0, Eq. (85) becomes invalid because of the constraint $t \gg \epsilon/a^3$. We adopt the conservative estimate a_L for a. Note that a_L is approximately linear in $(J_0 - J^*)$ for small $(J_0 - J^*)$ [Eq. (72)]. Taking the largest J^* consistent with $t \gg \epsilon/a^3$, we get

$$J_0 - J^* \approx \alpha_2 t^{-1/3} \tag{86}$$

which implies

$$\ln M(t) \leqslant -\lambda_0 t - \alpha_2' t^{1/3} (\ln t)^{2/3} \tag{87}$$

where α_2 and α_2' are some constants.

Both (82) and (87) have the same asymptotic behavior, and so we conclude that for large times t

$$\ln M(t) \approx -t(1 - \tanh 2\beta J_0) - ct^{1/3} (\ln t)^{2/3} \tag{88}$$

The behavior of the density of states $F(\lambda)$ for λ near λ_0 can be obtained by taking a Laplace transform of the above equation.

If $P(J)$ dies off faster than a power of $(J_0 - J)$ near $J = J_0$, the behavior of the magnetization is somewhat different. Consider, for instance, $P(J)$ such

that

$$\int_J^{J_0} dJ'\, P(J') \approx \exp[-K(J_0 - J)^{-b}] \tag{89}$$

where K and b are positive constants. Then for large times, $\Phi(t)$ decreases as $\exp(-t^{\nu})$ with

$$\nu = (1 + 2b)/(3 + 2b) \tag{90}$$

Other probability distributions can be treated similarly.

APPENDIX

In this appendix, we collect some of the evidence in favor of the conjectured ferromagnetic inequality of Section 5.

We first show that the $S_i(t)$ are positive, monotonically decreasing functions of time. Define $\bar{S}_i(t) = S_i(t)e^t$. Then it follows from Eq. (9) that

$$\frac{d}{dt}\bar{S}_i(t) = C_i^{+}\bar{S}_{i+1}(t) + C_i^{-}\bar{S}_{i-1}(t) \tag{A1}$$

Since $\bar{S}(t = 0) = 1$ for all i and $C_i^{\pm}$ are all positive for nonnegative $J_{i+1/2}$, it follows that

$$\frac{d}{dt}\bar{S}_i(t) \geqslant 0 \tag{A2}$$

This, in particular, implies that all $S_i(t)$ are positive and

$$S_i(t) \geqslant \exp(-t) \tag{A3}$$

Now, we show that dS_i/dt is nonpositive. Assume the contrary. Since at $t = 0$, dS_i/dt is negative for all i, there must exist an earliest time t_0 and some site j such that dS_j/dt changes sign at $t = t_0$ and the $S_i(t)$ are monotonically decreasing for all i up to time $t = t_0$. Then from Eq. (9)

$$S_j(t_0) = C_j^{+}S_{j+1}(t_0) + C_j^{-}S_{j-1}(t_0) \tag{A4}$$

But direct integration of Eq. (9) gives

$$S_j(t_0) = e^{-t} + e^{-t}\int_0^t dt'\, e^{t'}[C_j^{+}S_{j+1}(t') + C_j^{-}S_{j-1}(t')] \tag{A5}$$

But by assumption all $S_i(t)$ are monotonically decreasing up to time t_0, and hence for all $t' < t_0$

$$S_{j\pm1}(t') \geqslant S_{j+1}(t_0) \tag{A6}$$

Combining this with Eq. (A5), we get

$$S_j(t_0) \geqslant e^{-t_0} + (1 - e^{-t_0})[C_j^+ S_{j+1}(t_0) + C_j^- S_{j-1}(t_0)] \tag{A7}$$

But the right-hand side of the above inequality is strictly greater than $[C_j^+ S_{j+1}(t_0) + C_j^- S_{j-1}(t_0)]$. This contradicts Eq. (A4). Hence no such time t_0 can be found, and the $S_i(t)$ are monotonically decreasing functions of time for all times. Notice that this implies that for all times t

$$S_i(t) \geqslant C_i^+ S_{i+1}(t) + C_i^- S_{i-1}(t) \tag{A8}$$

Consider a chain C with bond strengths $\{J_{i+1/2}\}$. We choose some bond strength $J_{j+1/2}$ and increase it by a very small amount $\delta J_{j+1/2}$, keeping all other bond strengths unchanged. Let the new single spin expectation values be $S_i(t) + \delta S_i(t)$. The ferromagnetic inequality will be proved if we can show that $\delta S_i(t)$ is nonnegative for all i and all times t. To lowest order in $\delta J_{j+1/2}$, the $\delta S_i(t)$ satisfy the equations

$$(1 + d/dt)\, \delta S_i(t) = C_i^+ \, \delta S_{i+1}(t) + C_i^- \, \delta S_{i-1}(t), \qquad i \neq j \text{ or } j+1 \tag{A9}$$

For $i = j$ or $j + 1$, the equations are

$$\left(1 + \frac{d}{dt}\right) \delta S_i(t) = C_i^+ \, \delta S_{i+1}(t) + C_i^- \, \delta S_{i-1}(t) + \delta J_{j+1/2}\left[\left(\frac{\partial C_i^+}{\partial J_{j+1/2}}\right) S_{i+1}(t) + \left(\frac{\partial C_i^-}{\partial J_{j+1/2}}\right) S_{i-1}(t)\right] \tag{A10}$$

From the initial conditions, $\delta S_i(t = 0)$ are zero for all i. It is easy to show that for very short times, δS_{j+1} and δS_j are positive; and hence from Eq. (A9), all δS_i are positive. Hence the ferromagnetic inequality holds for sufficiently short times.

Let us assume that it is violated for the first time at $t = t_0$ for spin at some site k. Then $\delta S_k(t_0)$ is zero at $t = t_0$ and it crosses over to negative values for $t > t_0$. If $k \neq j$ or $j + 1$, we have the integral representation of δS_k [from Eq. (A9)]

$$\delta S_k(t_0) = \int_0^{t_0} dt' \exp(t' - t_0)[C_k^+ \, \delta S_{k+1}(t') + C_k^- \, \delta S_{k-1}(t')] \tag{A11}$$

This implies that $\delta S_k(t_0)$ is strictly greater than zero, and contradicts our assumption of $\delta S_k(t_0)$ being equal to zero. Hence we conclude that if the ferromagnetic inequality is violated, it must be violated at the sites j or $j + 1$ first, and only then can the affliction spread to more distant spins. The proof

of the inequality would be complete if we could show that $\delta S_j(t)$ and $\delta S_{j+1}(t)$ are never negative. Unfortunately, this last step is nontrivial due to the presence of the inhomogeneous term in Eq. (A10). It is easy to see that $\partial C_j^-/\partial J_{j+1/2}$ and $\partial C_{j+1}^+/\partial J_{j+1/2}$ are negative, and hence the sign of the inhomogeneous term in Eq. (A10) is indefinite. The sign depends on the relative magnitudes of S_{j-1} and S_{j+1} (say in the equation corresponding to δS_j), which in turn depend on magnitudes of couplings to other spins, etc.

However, we note that in the limit of very large temperatures $\beta \ll 1$, the hyperbolic tangents in $C_i^{\pm}$ can be replaced by their arguments. In that case $\partial C_i^{\pm}/\partial J_{j+1/2} \geqslant 0$ $(i = j, j+1)$, and the inhomogeneous term is always positive.

We have thus shown that in the case of a linear chain, the ferromagnetic inequality holds for very large temperatures at all times, and for all temperatures for very short times. For more general interactions $\{J_{ij}\}$, the equations of evolution of $S_i(t)$ involve higher order correlation functions, and the proof is a much harder proposition.

NOTE ADDED IN PROOF

Although we have explicitly considered only ferromagnetic distributions in this paper, our results also hold for bounded distributions with mixed ferromagnetic and antiferromagnetic bonds. However, the times at which the results are valid are larger than those considered in Refs. 12 and 13.

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